

# Regular and exact (virtual) double categories

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## Abstract

We propose definitions of regular and exact (virtual) double categories, proving a number of results which parallel many basic results in the theory of regular and exact categories. We show that any regular virtual double category admits a factorization system which generalizes the factorization of a functor between categories into a bijective-on-objects functor followed by a fully-faithful functor. Finally, we show that our definition of exact double category is equivalent to an axiom proposed by Wood, and very closely related to the “tight Kleisli objects” studied by Garner and Shulman.

## 1 Introduction

Category theory, besides having proven itself very generally useful, with examples of categories arising in most every branch of mathematics, has also proven itself very generalizable. For instance, enriched categories, internal categories, fibered categories, and quasicategories are some of the many variations and generalizations of the definition of category which have established themselves in modern mathematics.

Each of these has a theory which closely parallels that of ordinary categories, with functors and natural transformations, adjunctions, (weighted) limits and colimits, (pointwise) Kan extensions, the Yoneda lemma, and so on all playing central roles. It is natural to search for a common framework in which this body of definitions and results—which we refer to as *formal category theory*—can be developed once and specialized to each existing and future collection of “category-like structures”.

The obvious candidate for such a common framework is the theory of *2-categories*, or their less strict variation, *bicategories*. Every example of “category-like structures” can be assembled into a bicategory, so many people have tried to develop formal category theory at the level of generality of an arbitrary 2-category. However, it was quickly apparent that without more structure, important concepts like weighted limits and colimits and the Yoneda embedding do not have an adequate expression.

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One proposal for extra structure supporting a robust formal category theory was given by Wood in [7, 8]. The motivation for his proposal is that, besides functors and natural transformations, *profunctors* between categories are also a fundamental part of category theory (though often in the background). Wood defined an extra structure on a bicategory  $\mathcal{B}$ , together with a set of axioms, which “equips  $\mathcal{B}$  with abstract proarrows”. We will refer to this structure as a *proarrow equipment* for short.

In [5], Shulman showed that (pseudo) double categories satisfying a simple property are essentially equivalent to Wood’s proarrow equipments. Shulman called these double categories *framed bicategories*, though in [3] and elsewhere he has switched to referring to them simply as *equipments*, which we will do as well. He moreover demonstrated that the double category formulation of equipments makes clear the “right” definitions of functors and transformations, leading to a well-behaved 2-category of equipments.

In [3], Cruttwell and Shulman generalized equipments to *virtual equipments*, which are *virtual double categories* satisfying some simple properties. In a virtual equipment, composition of proarrows may not exist, yet there is still enough structure to support the development of formal category theory. They also show that all types of “generalized multicategory”, of which the majority of category-like structures are examples, arise as the objects in the virtual equipment of “monoids and modules” in some virtual equipment. Thus we can see that, just as most known types of algebraic or geometric structure can be assembled into a category, most known types of category-like structures (and more besides) can be assembled into a virtual equipment.

In classical category theory, there is a hierarchy of additional properties a category  $\mathcal{C}$  might have, beginning with  $\mathcal{C}$  simply having finite limits, and culminating with  $\mathcal{C}$  being a Grothendieck topos. The higher up this hierarchy  $\mathcal{C}$  is, the more “set-like” it is. Some of the intermediate levels in this hierarchy are regular, exact, coherent, and extensive categories, and pretoposes.

In this paper, we propose a beginning to an analogous hierarchy of additional properties on a virtual equipment. The higher up this hierarchy a virtual equipment  $\mathbb{D}$  is, the more properties it shares with categories and profunctors, and hence the more elements of formal category theory it should be possible to interpret inside  $\mathbb{D}$ . In particular, we propose in this paper definitions of *regular virtual equipment* and of *exact virtual equipment*.

In Section 2, we review the definitions of (virtual) double category and (virtual) equipment, as well as the construction of monoids and modules in a virtual double category. In Section 3, we define the “collapse” of a monoid, which plays a role in the theory analogous to coequalizers in the theory of regular and exact categories, and which is closely related to the Kleisli object of a monad.

Section 4 gives the definition of regular virtual equipment and proves some basic results which parallel the typical exposition of regular categories. In particular, we show that just as every regular category has a factorization system generalizing the epi/mono

image factorization in **Set**, every regular virtual equipment has a factorization system generalizing the bijective-on-objects/fully-faithful image factorization in **Cat**.

Lastly, Section 5 gives the definition of exact virtual equipment. The main result in this section is that exactness in our sense is essentially equivalent to Wood’s “Axiom 5” from [8]. This axiom, and the closely related “tight Kleisli objects” from [4], involves Kleisli objects and Eilenberg-Moore objects for monads in the bicategory of proarrows. While those papers clearly show that this is an important construction for formal category theory, it always felt to the author to be counter to Shulman’s “philosophy that the [proarrows] are not ‘morphisms’, but rather objects in their own right” [5]. The definition of exact virtual equipment gives an equivalent condition which we feel adheres to this philosophy, and establishes a tight analogy with a large body of classical category theory which we hope will stimulate further work in this direction.

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### Notational conventions

This paper deals with categories, 2-categories/bicategories, and (virtual) double categories, and so it is helpful to establish a notational convention to keep straight the various structures. In this paper, we write category variables  $\mathcal{C}$  in a caligraphic font (except when working inside the equipment **Prof**, where it would be distracting), while we write named categories such as **Set** and **Cat** in a bold roman font. 2-categories and bicategories such as **Cat** we write with a script-style first letter, and bicategory variables  $\mathcal{B}$  similarly. Double categories and virtual double categories we write with the first letter in a blackboard font:  $\mathbb{D}$ , **Prof**.

## 2 (Virtual) double categories and equipments

We begin by recalling some definitions from [5, 3] which are at the center of the present paper.

**Definition 2.1.** A *virtual double category*  $\mathbb{D}$  consists of the following data:

- A category  $\mathbb{D}_0$ , which we refer to as the *vertical category* of  $\mathbb{D}$ . For any two objects  $c, d \in \mathbb{D}_0$ , we will write  $\mathbb{D}(c, d) = \mathbb{D}_0(c, d)$  for the set of vertical arrows from  $c$  to  $d$ .
- For any two objects  $c, d \in \mathbb{D}_0$ , a set of horizontal arrows, which we refer to as proarrows and draw with a slash:  $c \rightarrowtail d$ .

- 2-cells, which have the shape

$$\begin{array}{ccccccc}
 c_0 & \xrightarrow{A_1} & c_1 & \xrightarrow{A_2} & c_2 & \xrightarrow{A_3} & \cdots & \xrightarrow{A_n} & c_n \\
 f \downarrow & & & & \Downarrow \phi & & & & \downarrow g \\
 d_0 & \xrightarrow{\quad\quad\quad} & & & B & \xrightarrow{\quad\quad\quad} & & & d_1
 \end{array} \tag{1}$$

for any  $n \geq 0$ . We will call  $f$  and  $g$  the *left frame* and *right frame* of  $\phi$ , and call the string  $A_1, \dots, A_n$  the (multi-)source and  $B$  the target of  $\phi$ . We will write  ${}_f\mathbb{D}_g(A_1, \dots, A_n; B)$  for the set of all cells of shape (1) in  $\mathbb{D}$ , and we write  $\mathbb{D}(A_1, \dots, A_n; B)$  for the set of cells with  $f$  and  $g$  identities.

- For each proarrow  $A: c \rightrightarrows d$  there is an identity 2-cell

$$\begin{array}{ccc}
 c & \xrightarrow{A} & d \\
 \parallel & \Downarrow 1_A & \parallel \\
 c & \xrightarrow{A} & d
 \end{array}$$

- Composition of 2-cells is like composition in a multicategory. So given the 2-cell  $\phi$  in (1) and  $n$  other 2-cells with horizontal targets  $A_1, \dots, A_n$ , there is a composite 2-cell with the evident shape. This composition operation satisfies unit and associativity axioms like in a multicategory.

We will now introduce the primary running examples of this paper.

*Example 2.2.* There is a virtual double category  $\mathbb{R}\mathbf{el}$  with vertical category  $\mathbb{R}\mathbf{el}_0 = \mathbf{Set}$ , and with proarrows  $R: a \rightrightarrows b$  given by relations  $R \subseteq b \times a$ . There is a 2-cell of the form (1) if and only if for every tuple  $(x_0, \dots, x_n) \in c_0 \times \cdots \times c_n$ , the implication

$$A_1(x_1, x_0) \wedge \cdots \wedge A_n(x_n, x_{n-1}) \Rightarrow B(g(x_n), f(x_0)).$$

holds.

*Example 2.3.* There is a virtual double category  $\mathbb{P}\mathbf{rof}$  with vertical category  $\mathbb{P}\mathbf{rof}_0 = \mathbf{Cat}$ , and with proarrows  $P: C \rightrightarrows D$  given by profunctors  $P: D^{\text{op}} \times C \rightarrow \mathbf{Set}$ . Given an element  $x \in P(d, c)$  and morphisms  $f: c \rightarrow c'$  in  $C$  and  $g: d' \rightarrow d$  in  $D$ , we will write the functorial action as  $P(g, f)(x) = f \cdot x \cdot g$ .

A 2-cell of the form

$$\begin{array}{ccccccc}
 C_0 & \xrightarrow{P_1} & C_1 & \xrightarrow{P_2} & C_2 & \xrightarrow{P_3} & \cdots & \xrightarrow{P_n} & C_n \\
 F \downarrow & & & & \Downarrow \phi & & & & \downarrow G \\
 D_0 & \xrightarrow{\quad\quad\quad} & & & Q & \xrightarrow{\quad\quad\quad} & & & D_1
 \end{array} \tag{2}$$

is a family of functions  $P_1(c_1, c_0) \times \cdots \times P_n(c_n, c_{n-1}) \rightarrow Q(Gc_n, Fc_0)$  for each tuple of objects  $(c_0, \dots, c_n) \in C_0 \times \cdots \times C_n$ , which is natural in each of the  $C_i$ . For  $C_0$ , naturality

means for each  $f: c_0 \rightarrow c'_0$  and each  $(x_1, \dots, x_n) \in P_1(c_1, c_0) \times \dots \times P_n(c_n, c_{n-1})$ , we have  $\phi(f \cdot x_1, x_1, \dots, x_n) = F(f) \cdot \phi(x_1, \dots, x_n)$ , while naturality in  $C_1$  means for each  $g: c_1 \rightarrow c'_1$  we have  $\phi(x_1 \cdot g, x_2, \dots, x_n) = \phi(x_1, g \cdot x_2, \dots, x_n)$ , and similarly for  $C_2, \dots, C_n$ .

**Definition 2.4.** Let  $\mathbb{D}$  be a virtual double category. The virtual double category  $\mathbf{Mod}(\mathbb{D})$  of *monoids and modules* is defined as follows:

- The objects are *monoids* in  $\mathbb{D}$ : tuples  $(c, M, e_M, m_M)$  consisting of an object  $c$  of  $\mathbb{D}$ , a proarrow  $M: c \rightarrowtail c$ , and unit and multiplication cells

$$\begin{array}{ccc} & c & \\ & \swarrow \quad \searrow & \\ c & \xrightarrow{M} & c \end{array} \quad \Downarrow e_M \quad \begin{array}{ccc} c & \xrightarrow{M} & c \\ \parallel & \Downarrow m_M & \parallel \\ c & \xrightarrow{M} & c \end{array}$$

satisfying the evident unit and associativity axioms.

- The vertical arrows are *monoid homomorphisms*: pairs  $(f, \vec{f})$  of a vertical arrow  $f: c \rightarrow d$  in  $\mathbb{D}$  and a cell

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ f \downarrow & \Downarrow \vec{f} & \downarrow f \\ d & \xrightarrow{N} & d \end{array}$$

which respects the unit and multiplication cells of  $M$  and  $N$ .

- The proarrows  $B: M \rightarrowtail N$  are *bimodules*: triples  $(B, l_B, r_B)$  consisting of a proarrow  $B: c \rightarrowtail d$  in  $\mathbb{D}$  and cells

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ \parallel & \Downarrow l_B & \parallel \\ c & \xrightarrow{B} & d \end{array} \quad \begin{array}{ccc} c & \xrightarrow{B} & d \\ \parallel & \Downarrow r_B & \parallel \\ c & \xrightarrow{B} & d \end{array}$$

satisfying evident monoid action axioms.

- The 2-cells are *bimodule homomorphisms*: cells in  $\mathbb{D}$

$$\begin{array}{ccccccc} c_0 & \xrightarrow{B_1} & c_1 & \xrightarrow{B_2} & \dots & \xrightarrow{B_n} & c_n \\ f \downarrow & & & \Downarrow \phi & & & \downarrow g \\ d_0 & \xrightarrow{A} & & & & & d_1 \end{array}$$

which are compatible with the left and right actions of the bimodules.

*Remark 2.5.* Given two monoids  $M: c \rightarrowtail c$  and  $N: d \rightarrowtail d$  in a virtual double category  $\mathbb{D}$ , we will write  ${}_M\mathbf{Bimod}_N$  for the *category* of  $(M, N)$ -bimodules, i.e. proarrows  $M \rightarrowtail N$  in  $\mathbf{Mod}(\mathbb{D})$ .

In a multicategory, tensor products of objects can be captured via a universal property. In this way, monoidal categories are equivalent to multicategories in which the tensor product of any list of objects exists. Similarly, composition of proarrows in a virtual double category can be captured by a universal property, and virtual double categories in which all such composites exist are equivalent to double categories. Note that in this paper, as in [5], double categories are always assumed to be *pseudo double categories*, in which composition in the vertical direction is strictly associative and unital, and in which composition in the horizontal direction is associative and unital only up to coherent isomorphism.

**Definition 2.6.** A cell

$$\begin{array}{c} \cdot \xrightarrow{P_1} \cdot \xrightarrow{P_2} \cdots \xrightarrow{P_n} \cdot \\ \parallel \quad \text{opcart} \quad \parallel \\ \cdot \xrightarrow{Q} \cdot \end{array} \quad (3)$$

in a virtual double category is said to be *opcartesian* if any cell

$$\begin{array}{c} \cdot \xrightarrow{R_1} \cdot \xrightarrow{R_2} \cdots \xrightarrow{R_m} \cdot \xrightarrow{P_1} \cdot \xrightarrow{P_2} \cdots \xrightarrow{P_n} \cdot \xrightarrow{S_1} \cdot \xrightarrow{S_2} \cdots \xrightarrow{S_k} \cdot \\ f \downarrow \quad \quad \quad \Downarrow \quad \quad \quad \downarrow g \\ \cdot \xrightarrow{T} \cdot \end{array}$$

factors through it uniquely as

$$\begin{array}{c} \cdot \xrightarrow{R_1} \cdot \xrightarrow{R_2} \cdots \xrightarrow{R_m} \cdot \xrightarrow{P_1} \cdot \xrightarrow{P_2} \cdots \xrightarrow{P_n} \cdot \xrightarrow{S_1} \cdot \xrightarrow{S_2} \cdots \xrightarrow{S_k} \cdot \\ \parallel \quad \parallel \quad \parallel \quad \text{opcart} \quad \parallel \quad \parallel \quad \parallel \\ \cdot \xrightarrow{R_1} \cdot \xrightarrow{R_2} \cdots \xrightarrow{R_m} \cdot \xrightarrow{Q} \cdot \xrightarrow{S_1} \cdot \xrightarrow{S_2} \cdots \xrightarrow{S_k} \cdot \\ f \downarrow \quad \quad \quad \Downarrow \quad \quad \quad \downarrow g \\ \cdot \xrightarrow{T} \cdot \end{array}$$

Thus a cell of the form (3) is opcartesian precisely if composition with it induces a bijection

$${}_f\mathbb{D}_g(R_1, \dots, R_m, Q, S_1, \dots, S_k; T) \cong {}_f\mathbb{D}_g(R_1, \dots, R_m, P_1, \dots, P_n, S_1, \dots, S_k; T)$$

for any  $f, g, T, R_1, \dots, R_m$ , and  $S_1, \dots, S_k$ .

Whenever an opcartesian cell (3) exists, we will refer to  $Q$  as *the composite* of the  $P_i$ 's, and write it as  $P_1 \odot \cdots \odot P_n$ . In the  $n = 0$  case, if there is an opcartesian cell of the form

$$\begin{array}{ccc} & c & \\ \swarrow & \Downarrow & \searrow \\ c & \xrightarrow{U_c} & c \end{array}$$

we say that  $c$  has a unit  $U_c$ . When clear from context, we will often write  $c$  for the unit proarrow  $U_c$ . Likewise, for any vertical arrow  $f: c \rightarrow d$  we will often write  $f$  for

the unit 2-cell

$$\begin{array}{ccc} c & \xrightarrow{\epsilon} & c \\ f \downarrow & \Downarrow f & \downarrow f \\ d & \xrightarrow[d]{} & d \end{array}$$

which is induced by  $f$  using the universal property of the units.

**Definition 2.7.** Say that a virtual double category  $\mathbb{D}$  *has units* if every object has a unit. Say that  $\mathbb{D}$  *has composites* if every string of  $n \geq 0$  composable proarrows has a composite.

**Definition 2.8.** If a virtual double category  $\mathbb{D}$  has units, then we can define a *vertical 2-category*  $\mathbf{Vert}(\mathbb{D})$ . The objects and morphisms of  $\mathbf{Vert}(\mathbb{D})$  are the objects and vertical arrows of  $\mathbb{D}$ , while for any pair of morphisms  $f, g: c \rightarrow d$ , the 2-cells  $\phi: f \Rightarrow g$  are defined to be 2-cells in  $\mathbb{D}$  of the form

$$\begin{array}{ccc} c & \xrightarrow{\epsilon} & c \\ g \downarrow & \Downarrow \phi & \downarrow f \\ d & \xrightarrow[d]{} & d \end{array}$$

If  $\mathbb{D}$  has *all* composites, then we can also define a *horizontal bicategory*  $\mathbf{HCor}(\mathbb{D})$ . The objects and morphisms of  $\mathbf{HCor}(\mathbb{D})$  are the objects and proarrows of  $\mathbb{D}$ , and the 2-cells are the 2-cells of  $\mathbb{D}$  with identity left and right frames. The bicategory axioms follow from the universal property of the composites.

Even if  $\mathbb{D}$  does not have all composites, we will sometimes abuse notation by writing  $\mathbf{HCor}(\mathbb{D})(c, d)$  for the category of proarrows  $c \twoheadrightarrow d$ .

*Example 2.9.* The virtual double category  $\mathbf{Rel}$  has composites. For any set  $A$ , the unit relation  $A: A \twoheadrightarrow A$  is simply the equality relation:  $A(a_1, a_2) \Leftrightarrow a_1 = a_2$ . For any composable pair of relations  $R: A \twoheadrightarrow B$ ,  $S: B \twoheadrightarrow C$ , the composite is the usual composition of relations:

$$(R \odot S)(c, a) \Leftrightarrow \exists b \in B. R(b, a) \wedge S(c, b)$$

*Example 2.10.* The virtual double category  $\mathbf{Prof}$  has composites as well. For any category  $C$ , the unit profunctor  $C: C \twoheadrightarrow C$  is the hom profunctor  $C^{\text{op}} \times C \rightarrow \mathbf{Set}$ , thus  $C(c_1, c_2) = \text{Hom}_C(c_1, c_2)$ . For any composable pair of profunctors  $P: C \twoheadrightarrow D$  and  $Q: D \twoheadrightarrow E$ , the composite can be defined as a coend

$$(P \odot Q)(e, c) = \int^{d \in D} P(d, c) \times Q(e, d).$$

This coend can be equivalently constructed as a quotient of  $\prod_{d \in D} (P(d, c) \times Q(e, d))$ , where for any  $f: d \rightarrow d'$  in  $D$  and any  $p \in P(d', c)$  and  $q \in Q(e, d)$ , we identify  $(p \cdot f, q)$  and  $(p, f \cdot q)$ . In this way, profunctor composition can be seen as analogous

to the tensor product of bimodules. This analogy between categories/profunctors and rings/bimodules is a very fruitful one, and in fact by generalizing to enriched categories, rings and bimodules can be seen as a special case of enriched categories and profunctors.

*Example 2.11.* For any virtual double category  $\mathbb{D}$ , the virtual double category  $\mathbf{Mod}(\mathbb{D})$  will always have units, though does not have all composites in general. For any monoid  $(c, M)$ , it is not hard to see that the unit bimodule is simply  $M : c \rightarrow c$  regarded as a  $(M, M)$ -bimodule. In [5] it is shown that if  $\mathbb{D}$  has composites, and has local reflexive coequalizers which are preserved under composition, then  $\mathbf{Mod}(\mathbb{D})$  has composites.

**Definition 2.12.** A cell

$$\begin{array}{ccc} \cdot & \xrightarrow{P} & \cdot \\ f \downarrow & \text{cart} & \downarrow g \\ \cdot & \xrightarrow{Q} & \cdot \end{array} \quad (4)$$

in a virtual double category is said to be *cartesian* if any cell

$$\begin{array}{ccccc} \cdot & \xrightarrow{R_1} & \cdot & \xrightarrow{R_2} & \dots & \xrightarrow{R_n} & \cdot \\ f \circ h \downarrow & & & \Downarrow & & & \downarrow g \circ k \\ \cdot & & & & & \xrightarrow{Q} & \cdot \end{array}$$

factors through it uniquely as

$$\begin{array}{ccccc} \cdot & \xrightarrow{R_1} & \cdot & \xrightarrow{R_2} & \dots & \xrightarrow{R_n} & \cdot \\ h \downarrow & & & \Downarrow & & & \downarrow k \\ \cdot & & & & & \xrightarrow{P} & \cdot \\ f \downarrow & & \text{cart} & & & & \downarrow g \\ \cdot & & & \xrightarrow{Q} & & & \cdot \end{array}$$

Thus a cell of the form (4) is cartesian precisely if composition with it induces a bijection

$${}_h\mathbb{D}_k(R_1, \dots, R_n; P) \cong {}_{fh}\mathbb{D}_{gk}(R_1, \dots, R_n; Q)$$

for any  $h, k$ , and  $R_1, \dots, R_n$ .

When a cartesian cell of the form (4) exists, we say that  $P$  is (isomorphic to) the *restriction* of  $Q$  along  $f$  and  $g$ , written  $Q(g, f)$ . We say that a virtual double category *has restrictions* if  $Q(g, f)$  exists for all compatible  $Q, f$ , and  $g$ .

**Definition 2.13.** A *virtual equipment*  $\mathbb{D}$  is a virtual double category which has units and restrictions. If  $\mathbb{D}$  has all composites, hence is a double category, we will call  $\mathbb{D}$  an *equipment* (called a *framed bicategory* in [5]).

*Example 2.14.*  $\mathbf{Rel}$  is an equipment: given functions  $f : A \rightarrow B$  and  $g : C \rightarrow D$ , and a relation  $R : B \rightarrow D$ , the restriction is given by  $R(g, f)(c, a) \Leftrightarrow R(g(c), f(a))$ .



$\mathbb{P}\mathbf{rof}$  is also an equipment: given functors  $F: A \rightarrow B$  and  $G: C \rightarrow D$ , and a profunctor  $P: B \nrightarrow D$ , the restriction is given by  $P(G, F)(c, a) = P(Gc, Fa)$ . In other words,  $P(G, F)$  is the composition

$$C^{\text{op}} \times A \xrightarrow{G^{\text{op}} \times F} D^{\text{op}} \times B \xrightarrow{P} \mathbf{Set}.$$

If  $\mathbb{D}$  is a virtual equipment, then so is  $\mathbf{Mod}(\mathbb{D})$ . See [3] for details.

*Example 2.15.* Let  $\mathcal{C}$  be a category with pullbacks. There is an equipment  $\mathbf{Span}(\mathcal{C})$  whose vertical category is  $\mathcal{C}$ , and whose proarrows  $S: c \nrightarrow d$  are spans  $d \leftarrow S \rightarrow c$ . Composition of spans is formed by pullback, and the 2-cells are the evident thing.

In [5, 3] it is shown that  $\mathbf{Mod}(\mathbf{Span}(\mathcal{C}))$  is the equipment of categories, functors, and profunctors *internal* to  $\mathcal{C}$ . In particular,  $\mathbf{Prof} = \mathbf{Mod}(\mathbf{Span}(\mathbf{Set}))$ .

*Remark 2.16.* Any vertical arrow  $f: c \rightarrow d$  in a virtual equipment gives rise to the two proarrows  $d(1, f): c \nrightarrow d$  and  $d(f, 1): d \nrightarrow c$ , formed by restricting the unit proarrow on  $d$  along  $f$  on one side and an identity on the other. We will call proarrows of this form *representable*.

Representable proarrows play a special role in the theory. For instance, in [5] it is shown that if a double category has restrictions of this special form, then it in fact has *all* restrictions. The same is not true for virtual double categories, but the following proposition shows that, assuming all restrictions exist, then all restrictions can be recovered by composition with representable proarrows. For this reason, representable proarrows are also often called *base change objects*.

*Example 2.17.* In  $\mathbf{Prof}$ , a profunctor  $P: 1 \nrightarrow C$  is precisely a presheaf on  $C$ , while a functor  $x: 1 \rightarrow C$  is just an object of  $C$ . In this case,  $P$  is representable by the functor  $x$  if  $P \cong C(1, x)$ , i.e. if for every object  $y \in C$ ,  $P(y) \cong C(y, x)$ . This is the motivation for the term *representable profunctor*.

**Proposition 2.18.** *Let  $P: c \nrightarrow d$  be a proarrow and  $f: a \rightarrow c$  and  $g: b \rightarrow d$  be vertical arrows in a virtual equipment. Then the composite  $C(1, f) \odot P \odot B(g, 1)$  exists and is isomorphic to  $P(g, f)$ .*

*Moreover, there is a bijection between cells of the form*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 a_0 & \xrightarrow{P} & b_0 \\
 f_1 \downarrow & & \downarrow g_1 \\
 a_1 & & b_1 \\
 f_2 \downarrow & \Downarrow & \downarrow g_2 \\
 a_2 & & b_2 \\
 f_3 \downarrow & & \downarrow g_3 \\
 a_3 & \xrightarrow{Q} & b_3
 \end{array}
 & \text{and} &
 \begin{array}{ccccc}
 a_1 & \xrightarrow{a_1(f_1, 1)} & a_0 & \xrightarrow{P} & b_0 & \xrightarrow{b_1(1, g_1)} & b_1 \\
 f_2 \downarrow & & & \Downarrow & & & \downarrow g_2 \\
 a_2 & \xrightarrow{a_3(1, f_3)} & a_3 & \xrightarrow{Q} & b_3 & \xrightarrow{b_3(g_3, 1)} & b_2
 \end{array}
 \end{array}$$

Note that we can make sense of the cell on the right because the composition of the proarrows along the bottom exists (and is isomorphic to  $Q(g_3, f_3)$ ). We draw it this way to make the symmetry clear.

### 3 Collapse

In this section we will introduce a central concept of this paper: the collapse of a monoid or bimodule in a virtual equipment. This can be seen as a generalization both of the Kleisli object of a monad in a bicategory, and of the quotient of a relation in a category. It is essentially the same as the “tight Kleisli objects” considered in [4], though they worked in a slightly more general context.

**Definition 3.1.** An *embedding* of a monoid  $M: c \rightrightarrows c$  in a virtual equipment into an object  $x$  is a monoid homomorphism (Definition 2.4)  $(f, \vec{f})$  from  $M$  to the trivial monoid on  $x$ :

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ f \downarrow & \Downarrow \vec{f} & \downarrow f \\ x & \xrightarrow{x} & x \end{array}$$

We will sometimes write an embedding as  $(f, \vec{f}): (c, M) \rightarrow x$ , or even just  $f: M \rightarrow x$  when clear from context. We will write  $\text{Emb}(M, x)$  for the set of embeddings from  $M$  to  $x$ .

Likewise, an *embedding* of a  $(M, N)$ -bimodule  $B$  into a proarrow  $P: x \rightrightarrows y$  consists of monoid embeddings  $f: M \rightarrow x$  and  $g: N \rightarrow y$ , and a bimodule homomorphism from  $B$  to  $P$ , regarding  $P$  as a bimodule between the trivial monoids on  $x$  and  $y$ :

$$\begin{array}{ccc} c & \xrightarrow{B} & d \\ f \downarrow & \Downarrow \phi & \downarrow g \\ x & \xrightarrow{P} & y \end{array} \quad (5)$$

We will sometimes write such a bimodule embedding as  $f\phi_g: {}_M B_N \rightarrow P$ , and we will write  ${}_f \text{Emb}_g(B, P)$  for the set of all such embeddings, for fixed embeddings  $f: M \rightarrow x$  and  $g: N \rightarrow y$ .

Say an embedding (5) is *cartesian* if  $\vec{f}$ ,  $\vec{g}$ , and  $\phi$  are all cartesian cells.

*Example 3.2.* A monoid  $R: a \rightrightarrows a$  in  $\mathbb{R}\text{el}$  is precisely a reflexive transitive relation on the set  $a$ . An embedding  $(f, \vec{f}): R \rightarrow x$  is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\vec{f}} & x \\ \downarrow & & \downarrow \Delta \\ a \times a & \xrightarrow{f \times f} & x \times x \end{array}$$

or equivalently, a “fork”

$$R \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} a \xrightarrow{f} x$$

i.e. a function  $f: a \rightarrow x$  such that  $fp_1 = fp_2 (= \vec{f})$ .

We leave the proof of the following easy observation to the reader.

**Lemma 3.3.** *Given embeddings  $f: M \rightarrow x$  and  $g: N \rightarrow y$  in a virtual equipment  $\mathbb{D}$ , and a cartesian cell*

$$\begin{array}{ccc} c & \xrightarrow{P(g,f)} & d \\ f \downarrow & \Downarrow \phi & \downarrow g \\ x & \xrightarrow{P} & y \end{array}$$

*there is a unique  $(M, N)$ -bimodule structure on  $P(g, f)$  making  $\phi$  an embedding.*

*For any  $B \in {}_M\mathbf{Bimod}_N$  and any proarrow  $P \in \mathbf{HCor}(\mathbb{D})(x, y)$ , this construction induces a bijection  ${}_f\mathbf{Emb}_g(B, P) \cong {}_M\mathbf{Bimod}_N(B, P(g, f))$ , which is natural in  $B$  and  $P$ .*

**Definition 3.4.** Lemma 3.3 determines a functor

$${}_f\mathbf{Res}_g: \mathbf{HCor}(\mathbb{D})(x, y) \rightarrow {}_M\mathbf{Bimod}_N$$

for any pair of embeddings  $f: M \rightarrow x$  and  $g: N \rightarrow y$ . Thus for any  $P: x \rightarrow y$ ,  ${}_f\mathbf{Res}_g(P)$  is defined to be  $P(g, f)$  with the unique  $(M, N)$ -bimodule structure making the cartesian cell an embedding.

**Definition 3.5.** Let  $M: c \rightarrow c$  be a monoid in a virtual equipment  $\mathbb{D}$ . A *collapse* of  $M$  is a universal embedding of  $M$ . That is, a collapse of  $M$  is an object  $\langle M \rangle$  together with an embedding

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ i_M \downarrow & \Downarrow \vec{i}_M & \downarrow i_M \\ \langle M \rangle & \xrightarrow{\langle M \rangle} & \langle M \rangle \end{array} \quad (6)$$

such that any other embedding factors uniquely through  $\vec{i}_M$ :

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ f \downarrow & \Downarrow \vec{f} & \downarrow f \\ x & \xrightarrow{x} & x \end{array} = \begin{array}{ccc} c & \xrightarrow{M} & c \\ i_M \downarrow & \Downarrow \vec{i}_M & \downarrow i_M \\ \langle M \rangle & \xrightarrow{\langle M \rangle} & \langle M \rangle \\ \tilde{f} \downarrow & \Downarrow \tilde{f} & \downarrow \tilde{f} \\ x & \xrightarrow{x} & x \end{array}$$

In other words,  $\langle M \rangle$  represents the functor  $\mathbb{D}_0 \rightarrow \mathbf{Set}$  sending  $x$  to  $\mathbf{Emb}(M, x)$ .

*Example 3.6.* Given a monoid  $R: a \rightarrowtail a$  in  $\mathbf{Rel}$ , i.e. a (reflexive, transitive) relation on  $a$ , the collapse of  $R$  is a universal fork  $R \rightrightarrows a \rightarrow x$ , that is, a coequalizer of  $(p_1, p_2)$ .

*Example 3.7.* A monoid  $M: C \rightarrowtail C$  in  $\mathbf{Prof}$  is a profunctor  $M: C^{\text{op}} \times C \rightarrow \mathbf{Set}$  with a unit and multiplication. The unit amounts to a function from morphisms  $f: c \rightarrow d$  in  $C$  to elements  $e(f) \in M(c, d)$  of  $M$ , which is compatible with the functorial action on  $M$  in that  $h \cdot e(f) = e(h \circ f)$  and  $e(f) \cdot g = e(f \circ g)$  whenever these make sense.

The multiplication is an operation which, given elements  $m_1 \in M(c, d)$  and  $m_2 \in M(d, e)$ , assigns an element  $m_2 \bullet m_1 \in M(c, e)$ . This operation must be compatible with the functorial action, meaning  $(f \cdot m_2) \bullet m_1 = f \cdot (m_2 \bullet m_1)$ ,  $(m_2 \cdot g) \bullet m_1 = m_2 \bullet (g \cdot m_1)$ , and  $m_2 \bullet (m_1 \cdot h) = (m_2 \bullet m_1) \cdot h$ , whenever these make sense, and it must satisfy unit and associativity axioms:  $e(f) \bullet m = f \cdot m$ ,  $m \bullet e(g) = m \cdot g$ , and  $(m_3 \bullet m_2) \bullet m_1 = m_3 \bullet (m_2 \bullet m_1)$ .

The collapse of a monoid  $M$  is a category  $\langle M \rangle$  with the objects of  $C$ , and with hom sets  $\text{Hom}_{\langle M \rangle}(c, d) = M(c, d)$ . The multiplication of  $M$  defines the composition of  $\langle M \rangle$ , while the functorial action of  $C$  on  $M$  defines the identity-on-objects functor  $i_M: C \rightarrow \langle M \rangle$ .

*Example 3.8.* More generally (see Example 2.15), we can form the collapse of any monoid in  $\mathbf{Mod}(\mathbb{D})$ , for any virtual equipment  $\mathbb{D}$ . We will sketch how this works, leaving the routine verifications to the reader.

Suppose  $M: c \rightarrowtail c$  is a monoid in  $\mathbb{D}$ , i.e. an object  $(c, M) \in \mathbf{Mod}(\mathbb{D})$ , and let  $N: (c, M) \rightarrowtail (c, M)$  be a monoid in  $\mathbf{Mod}(\mathbb{D})$ . This means  $N$  is a  $(M, M)$ -bimodule, together with unit and multiplication bimodule homomorphisms.

The collapse of  $N$  will be  $N$  itself, forgetting the bimodule structure, but remembering the monoid structure. In particular, the unit of the collapse is the composition of the unit  $e_M: c \rightarrow M$  of  $M$  and the unit  $\eta_N: M \rightarrow N$  of  $N$ , while the multiplication of the collapse is simply the multiplication of  $N$ .

The collapse map  $i: (c, M) \rightarrow (c, N)$  is the unit  $\eta_N: M \rightarrow N$ , and  $\vec{i}$  is the identity on  $N$ .

**Definition 3.9.** Let  $M: c \rightarrowtail c$  and  $N: d \rightarrowtail d$  be monoids in a virtual equipment such that the collapses  $\langle M \rangle$  and  $\langle N \rangle$  exist, and let  $B: c \rightarrowtail d$  be a  $(M, N)$ -bimodule. A *collapse* of  $B$  is a universal embedding of  $B$ : a proarrow  $\langle B \rangle: \langle M \rangle \rightarrowtail \langle N \rangle$  together with an embedding

$$\begin{array}{ccc} c & \xrightarrow{B} & d \\ i_M \downarrow & \Downarrow i_B & \downarrow i_N \\ \langle M \rangle & \xrightarrow{\langle B \rangle} & \langle N \rangle \end{array} \quad (7)$$

such that any other embedding factors uniquely through  $i_B$ :

$$\begin{array}{ccc}
 c & \xrightarrow{B} & d \\
 f \downarrow & \Downarrow \phi & \downarrow g \\
 x & \xrightarrow{P} & y
 \end{array}
 =
 \begin{array}{ccc}
 c & \xrightarrow{B} & d \\
 i_M \downarrow & \Downarrow i_B & \downarrow i_N \\
 \langle M \rangle & \xrightarrow{\langle B \rangle} & \langle N \rangle \\
 \tilde{f} \downarrow & \Downarrow \tilde{\phi} & \downarrow \tilde{g} \\
 x & \xrightarrow{P} & y
 \end{array}
 \quad (8)$$

In other words, composition with  $i_B$  induces a bijection

$$\tilde{f}\mathbb{D}_{\tilde{g}}(\langle B \rangle; -) \cong \tilde{f}_{i_M} \text{Emb}_{\tilde{g}i_N}(B, -).$$

*Remark 3.10.* When it is not clear from context, we will speak of a “monoid collapse” or a “bimodule collapse” to specify which of Definitions 3.5 or 3.9 is meant.

**Proposition 3.11.** *Consider a cell in a virtual double category*

$$\begin{array}{ccc}
 c & \xrightarrow{B} & d \\
 i_M \downarrow & \Downarrow \phi & \downarrow i_N \\
 \langle M \rangle & \xrightarrow{P} & \langle N \rangle
 \end{array}$$

where  $B$  is a  $(M, N)$ -bimodule,  $i_M: M \rightarrow \langle M \rangle$  and  $i_N: N \rightarrow \langle N \rangle$  are collapse embeddings, and  $i_M \phi_{i_N}: {}_M B_N \rightarrow P$  is a bimodule embedding. The following are equivalent:

1. The embedding  $\phi$  is a bimodule collapse.
2. Composition with  $\phi$  induces a bijection  $\mathbb{D}(P; -) \cong {}_{i_M} \text{Emb}_{i_N}(B, -)$
3. Composition with  $\phi$  induces a bijection  $\mathbb{D}(P; -) \cong {}_M \text{Bimod}_N(B, {}_{i_M} \text{Res}_{i_N}(-))$ .

*Proof.* 2 and 3 are clearly equivalent by Lemma 3.3, and 2 easily follows from 1.

To see  $2 \Rightarrow 1$ , we have the chain of equivalences, for any  $f: \langle M \rangle \rightarrow x$ ,  $g: \langle N \rangle \rightarrow y$ , and  $Q: x \rightrightarrows y$ ,

$${}_f \mathbb{D}_g(P; Q) \cong \mathbb{D}(P; Q(g, f)) \cong {}_{i_M} \text{Emb}_{i_N}(B, Q(g, f)) \cong {}_{f i_M} \text{Emb}_{g i_N}(B, Q)$$

□

**Definition 3.12.** We will call the diagram (6) a *normal collapse* if it also exhibits  $\langle M \rangle$  as the bimodule collapse of the unit  $(M, M)$ -bimodule  $M: c \rightrightarrows c$ .

## 4 Regular virtual double categories

Let  $\mathcal{C}$  be a category with finite limits, and let  $f: c \rightarrow d$  be a morphism in  $\mathcal{C}$ . Recall the following standard definitions (see e.g. [1], [2]):

- The *kernel pair* of  $f$  is the pair  $p_1, p_2: R \rightrightarrows c$  given by the pullback

$$\begin{array}{ccc} R & \xrightarrow{p_1} & c \\ p_2 \downarrow & \lrcorner & \downarrow f \\ c & \xrightarrow{f} & d \end{array}$$

A kernel pair is always an internal equivalence relation: that is  $(p_1, p_2): R \rightarrow c \times c$  is a monomorphism ( $R$  is a relation), there exists a common section  $c \rightarrow R$  of  $p_1$  and  $p_2$  ( $R$  is reflexive), and  $R$  is similarly transitive and symmetric.

- An equivalence relation  $p_1, p_2: R \rightrightarrows c$  is called *effective* if it is the kernel pair of some morphism.
- $f$  is a *regular epimorphism* if it is the coequalizer of some parallel pair of arrows.
- $\mathcal{C}$  is a *regular category* if every effective equivalence relation has a coequalizer, and if regular epimorphisms are stable under pullback.

In a regular category  $\mathcal{C}$ , any morphism factors uniquely as a regular epimorphism followed by a monomorphism. In fact, a category is regular if and only if it has a such a factorization system and the regular epimorphisms are stable under pullback. In that way, regular categories are precisely those with “well-behaved” image factorizations.

Another common description of regular categories is that they are precisely those with a “good” theory of internal relations. In particular, we have the following construction.

**Definition 4.1.** For any regular category  $\mathcal{C}$ , we can define an equipment  $\mathbf{Rel}(\mathcal{C})$  as follows:

- The vertical category of  $\mathbf{Rel}(\mathcal{C})$  is  $\mathcal{C}$ .
- Proarrows  $R: a \rightarrowtail b$  are relations, i.e. monomorphisms  $R \hookrightarrow b \times a$ .
- 2-cells

$$\begin{array}{ccc} a & \xrightarrow{R} & b \\ f \downarrow & \Downarrow \phi & \downarrow g \\ c & \xrightarrow{S} & d \end{array} \quad (9)$$

are commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow & & \downarrow \\ b \times a & \xrightarrow{g \times f} & d \times c \end{array} \quad (10)$$

In particular, note that for any square of shape (9), there is at most one 2-cell  $\phi$  of that shape. We say that  $\mathbf{Rel}(\mathcal{C})$  is “locally posetal”.

The 2-cell (9) is cartesian if and only if (10) is a pullback.

- The unit relation  $a \rightarrowtail a$  is the diagonal  $\Delta: a \hookrightarrow a \times a$ . The composition  $R \odot S$  of two relations  $R: a \rightarrowtail b$  and  $S: b \rightarrowtail c$  is formed using pullbacks and the epi-mono factorization, as follows:

$$\begin{array}{ccccc}
 R \odot S & \xleftarrow{\quad} & \cdot & \xrightarrow{\quad} & R \times S \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 c \times a & \xleftarrow{\quad \pi \quad} & c \times b \times a & \xrightarrow{1 \times \Delta \times 1} & c \times b \times b \times a
 \end{array}$$

In this section, we will propose a definition of *regular virtual equipment*. In 4.1 we begin with the definition and some preliminary results and examples, and in 4.2 we give a generalization of the epi-mono factorization present in any regular category.

## 4.1 Definition and basic properties

**Definition 4.2.** Let  $f: c \rightarrow d$  be a vertical arrow in a virtual equipment. The *kernel* of  $f$  is defined to be the monoid obtained by restricting the trivial monoid on  $d$ :

$$\begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 f \downarrow & \text{cart} & \downarrow f \\
 d & \xrightarrow{d} & d
 \end{array} \quad (11)$$

So the multiplication  $\mu$  is the unique 2-cell satisfying

$$\begin{array}{ccc}
 \begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 \parallel & & \parallel \\
 c & \xrightarrow{\ker(f)} & c \\
 f \downarrow & & \downarrow f \\
 d & \xrightarrow{d} & d
 \end{array} & \xrightarrow{\Downarrow \mu} & \begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 \parallel & & \parallel \\
 c & \xrightarrow{\ker(f)} & c \\
 f \downarrow & & \downarrow f \\
 d & \xrightarrow{d} & d
 \end{array} \\
 \parallel & & \parallel \\
 d & \xrightarrow{d} & d
 \end{array} = \begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 f \downarrow & \text{cart} & \downarrow f \\
 d & \xrightarrow{d} & d \\
 \parallel & & \parallel \\
 d & \xrightarrow{d} & d
 \end{array} \quad (12)$$

and similarly for the unit  $\eta$ .

**Definition 4.3.** Let  $f: c \rightarrow d$  be a vertical arrow in a virtual equipment. Say that  $f$  is an *inclusion* if the unit 2-cell on  $f$  is cartesian, or equivalently if  $\ker(f)$  is the trivial monoid on  $c$ . We will denote inclusions by  $f: c \rightarrowtail d$ .

**Definition 4.4.** Say that a monoid  $M: c \rightarrowtail c$  in a virtual equipment is *effective* if  $M$  is the kernel of some vertical arrow.

Similarly, say that a  $(M, N)$ -bimodule  $B: c \rightarrowtail d$  is effective if  $M$  and  $N$  are effective, with  $M \cong \ker(f)$  and  $N \cong \ker(g)$  for some  $f: c \rightarrow c'$  and  $g: d \rightarrow d'$ , and there exists a proarrow  $P: c' \rightarrowtail d'$  such that  $B \cong {}_f\text{Res}_g(P)$ . Equivalently,  $B$  is effective if there exists a cartesian embedding  $B \rightarrow P$  for some  $P$ .

**Definition 4.5.** Let  $f: c \rightarrow d$  be a vertical arrow in a virtual equipment. Say that  $f$  is a *regular cover* if the restriction (11) is a normal collapse cell. We will denote regular covers by  $f: c \twoheadrightarrow d$ .

*Example 4.6.* The inclusions in  $\mathbf{Rel}(\mathcal{C})$  are precisely the monomorphisms of  $\mathcal{C}$ , and the regular covers are the regular epimorphisms.

The inclusions in  $\mathbf{Prof}$  are the fully-faithful functors, and the regular covers are those functors which are bijective on objects.

**Definition 4.7.** Say that a virtual equipment  $\mathbb{D}$  is *regular* if

1. every effective monoid has a normal collapse,
2. for every proarrow  $B: d \twoheadrightarrow d'$  and regular covers  $f: c \twoheadrightarrow d$  and  $g: c' \twoheadrightarrow d'$ , the cartesian embedding

$$\begin{array}{ccc} c & \xrightarrow{f \text{ Res}_g(B)} & c' \\ f \downarrow & \text{cart} & \downarrow g \\ d & \xrightarrow{B} & d' \end{array}$$

is a bimodule collapse cell.

The equipments  $\mathbf{Rel}$  and  $\mathbf{Prof}$  are both regular. In fact, the next two propositions show that most “Rel-like” and “Prof-like” (virtual) equipments will be regular. (See [3] for an exhibition of some of the many examples of familiar structures arising as  $\mathbf{Mod}(\mathbb{D})$  for some  $\mathbb{D}$ .)

**Proposition 4.8.** *For any regular category  $\mathcal{C}$ , the equipment  $\mathbf{Rel}(\mathcal{C})$  is regular.*

*Proof.* The kernel of any vertical morphism  $f: a \twoheadrightarrow b$  is precisely the kernel pair of  $f$ , considered as an internal reflexive transitive relation  $\ker(f): a \twoheadrightarrow a$ . The collapse of  $\ker(f)$  exists because  $\mathcal{C}$  has coequalizers of kernel pairs.

It is not hard to check that a 2-cell (10) is a bimodule collapse if and only if  $f$ ,  $g$ , and  $\phi$  are all regular epimorphisms (hint: use the orthogonality of monos and regular epis).

Suppose  $R: a \twoheadrightarrow a$  is an effective monoid/relation, with collapse/coequalizer  $i: a \rightarrow \langle R \rangle$ . Then in the collapse cell

$$\begin{array}{ccc} M & \xrightarrow{\vec{i}} & \langle M \rangle \\ \downarrow & & \downarrow \Delta \\ a \times a & \xrightarrow{i \times i} & \langle M \rangle \times \langle M \rangle \end{array}$$

we can see that  $\vec{i}$  is a regular epimorphism as follows:  $p_1$  and  $p_2$  are split epis since  $R$  is reflexive, and in a regular category every split epi is a regular epi;  $i$  is a regular epi by definition; and  $\vec{i} = ip_1 (= ip_2)$  is a regular epi because regular epis are closed under composition. Hence the collapse is normal.



Finally, part 2 of Definition 4.7 follows because regular epis are closed under product and pullback.  $\square$

**Proposition 4.9.** *For any virtual equipment  $\mathbb{D}$ , the virtual equipment  $\mathbf{Mod}(\mathbb{D})$  is regular.*

*Proof.* We will provide a sketch, leaving the many straightforward but tedious verifications to the reader.

We saw in Example 3.8 that in fact *every* monoid  $N: (c, M) \twoheadrightarrow (c, M)$  in  $\mathbf{Mod}(\mathbb{D})$  has a collapse. It is not hard to see that  $\vec{i}_M$  is a cartesian cell, as its underlying cell in  $\mathbb{D}$  is the identity on  $N$ . We will see in Lemma 4.11 that  $\vec{i}_M$  being cartesian implies that the collapse is normal. Thus axiom 1 holds.

To verify axiom 2, we claim that a vertical morphism  $(f, \vec{f}): (c, M) \rightarrow (d, N)$  is a regular cover if and only if  $f$  is an isomorphism, and that for any regular covers  $f$  and  $g$ , a 2-cell  $\phi \in {}_f\mathbf{Mod}(\mathbb{D})_g(B, B')$  is a bimodule collapse if and only if it is cartesian, if and only if the underlying 2-cell in  $\mathbb{D}$  is an isomorphism.  $\square$

**Proposition 4.10.** *Condition 2 of Definition 4.7 is equivalent to the following:*

*2'. for every pair of regular covers  $f: c \twoheadrightarrow d$  and  $g: c' \twoheadrightarrow d'$ , the functor*

$${}_f\mathrm{Res}_g: \mathbf{HCor}(\mathbb{D})(d, d') \rightarrow {}_{\ker(f)}\mathrm{Bimod}_{{\ker(g)}}$$

*is fully-faithful.*

*Proof.* Let  $B: d \twoheadrightarrow d'$  be a proarrow. By Proposition 3.11, the embedding  ${}_f\mathrm{Res}_g(B) \rightarrow B$  is a collapse if and only if the function  $\mathbb{D}(B; -) \rightarrow {}_{\ker(f)}\mathrm{Bimod}_{{\ker(g)}}({}_f\mathrm{Res}_g(B), {}_f\mathrm{Res}_g(-))$  is a bijection.  $\square$

**Lemma 4.11.** *Let  $\mathbb{D}$  be a virtual equipment satisfying condition 2 of Definition 4.7. A collapse cell*

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ i_M \downarrow & \Downarrow \vec{i}_M & \downarrow i_M \\ \langle M \rangle & \xrightarrow{\langle M \rangle} & \langle M \rangle \end{array}$$

*in  $\mathbb{D}$  is normal if and only if  $\vec{i}$  is cartesian.*

*Proof.* Consider the diagram (letting  $i := i_M$ )

$$\begin{array}{ccc} \mathbb{D}(\langle M \rangle, -) & \xrightarrow[\cong]{i\mathrm{Res}_i} & {}_M\mathrm{Bimod}_M(i\mathrm{Res}_i(\langle M \rangle), i\mathrm{Res}_i(-)) \\ & \searrow 1 & \swarrow 2 \\ & {}_M\mathrm{Bimod}_M(M, i\mathrm{Res}_i(-)) & \end{array}$$

in which the two downwards functions are induced by composition with  $\vec{i}$ . The top function is a bijection by Proposition 4.10. By Proposition 3.11, the collapse is normal

precisely if 1 is a bijection. On the other hand, 2 is a bijection if and only if  $\vec{\tau}$  induces an isomorphism of bimodules  $M \cong {}_i\text{Res}_i(\langle M \rangle)$ , which by Lemma 3.3 happens if and only if  $\vec{\tau}$  is cartesian. Thus the collapse is normal if and only if  $\vec{\tau}$  is cartesian.  $\square$

**Corollary 4.12.** *In the presence of 2, condition 1 of Definition 4.7 is equivalent to:*

*1'. Every effective monoid  $M$  has a collapse  $(i, \vec{\tau})$ , and  $\vec{\tau}$  is cartesian.*

*In other words, in a regular virtual equipment, every kernel is the kernel of its collapse.*

**Proposition 4.13.** *Any effective bimodule in a regular virtual equipment has a collapse, and moreover the collapse cell is cartesian.*

*Proof.* Let  $M$  and  $N$  be effective monoids, and suppose given an effective  $(M, N)$ -bimodule  $B: c \rightarrowtail c'$ , with cartesian embedding

$$\begin{array}{ccc} c & \xrightarrow{B} & c' \\ f \downarrow & \text{cart} & \downarrow g \\ d & \xrightarrow{p} & d' \end{array} \quad (13)$$

We can factor this as

$$\begin{array}{ccc} c & \xrightarrow{B} & c' \\ i_M \downarrow & \Downarrow \phi & \downarrow i_N \\ \langle M \rangle & \xrightarrow{P(\tilde{g}, \tilde{f})} & \langle N \rangle \\ \tilde{f} \downarrow & \text{cart} & \downarrow \tilde{g} \\ d & \xrightarrow{p} & d' \end{array}$$

It follows that  $\phi$  is cartesian, hence a bimodule collapse (noting that  $\ker(i_M) \cong M$  by Corollary 4.12, and similarly for  $g$ ).  $\square$

**Lemma 4.14.** *Consider a diagram in a regular virtual equipment  $\mathbb{D}$  of the form*

$$\begin{array}{ccc} c & \xrightarrow{X} & c' \\ f \downarrow & \text{cart} & \downarrow f' \\ d & \xrightarrow{Y} & d' \\ g \downarrow & \Downarrow \phi & \downarrow g' \\ e & \xrightarrow{Z} & e' \end{array} \quad (14)$$

*in which  $f$  and  $f'$  are regular covers. If the composite 2-cell is cartesian, then  $\phi$  is also cartesian.*

*Proof.* We could also factor the composite 2-cell through the restriction  $Z(g', g)$ :

$$\begin{array}{ccc}
 c & \xrightarrow{X} & c' \\
 f \downarrow & \Downarrow \psi & \downarrow f' \\
 d & \xrightarrow{Z(g', g)} & d' \\
 g \downarrow & \text{cart} & \downarrow g' \\
 e & \xrightarrow{Z} & e'
 \end{array}$$

If the composite is cartesian, then so is  $\psi$ , hence by part 2 of Definition 4.7  $\psi$  is also a bimodule collapse cell. The upper 2-cell in (14) is a bimodule collapse cell for the same reason. But then by the universal property of bimodule collapse, this factorization is in fact isomorphic to (14), hence  $\phi$  is cartesian.  $\square$

## 4.2 The factorization system

One of the primary facts about any regular category is the existence of an image factorization. In a regular category  $\mathcal{C}$  there is an orthogonal factorization system  $(\mathcal{E}, \mathcal{M})$  where  $\mathcal{E}$  is the class of regular epimorphisms and  $\mathcal{M}$  is the class of monomorphisms. We will now see that a regular virtual equipment admits an analogous orthogonal factorization system.

In a regular category, the image of a morphism is defined to be the coequalizer of its kernel, and it is shown that any morphism factors through its image. We can perform the analogous construction in a regular virtual equipment: for any vertical arrow  $f: c \rightarrow d$  we define its image to be the collapse  $\langle \ker(f) \rangle$  of its kernel, and we get a unique arrow  $\tilde{f}: \langle \ker(f) \rangle \rightarrow d$  such that

$$\begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 i \downarrow & \Downarrow \tilde{i} & \downarrow i \\
 \langle \ker(f) \rangle & \xrightarrow{\langle \ker(f) \rangle} & \langle \ker(f) \rangle \\
 \tilde{f} \downarrow & \Downarrow \tilde{f} & \downarrow \tilde{f} \\
 d & \xrightarrow{d} & d
 \end{array}
 =
 \begin{array}{ccc}
 c & \xrightarrow{\ker(f)} & c \\
 f \downarrow & \text{cart} & \downarrow f \\
 d & \xrightarrow{d} & d
 \end{array}
 \quad (15)$$

Another standard fact from the theory of regular categories is that the classes of regular epimorphisms and strong epimorphisms coincide, where a morphism  $f$  is called a strong epimorphism if it is left-orthogonal to the class of monomorphisms. We begin with an analogous definition in the setting of virtual equipments.

**Definition 4.15.** Let  $f: a \rightarrow b$  be a vertical arrow in a virtual equipment  $\mathbb{D}$ . Say that  $f$  is a *strong cover* if it is left 2-orthogonal to the class of inclusions in the vertical

2-category  $\mathbf{Vert}(\mathbb{D})$ , i.e. if for any inclusion  $g: c \rightarrowtail d$  the commuting square

$$\begin{array}{ccc} \mathbf{Vert}(\mathbb{D})(b, c) & \xrightarrow{f \circ -} & \mathbf{Vert}(\mathbb{D})(a, c) \\ \downarrow - \circ g & \lrcorner & \downarrow - \circ g \\ \mathbf{Vert}(\mathbb{D})(b, d) & \xrightarrow{f \circ -} & \mathbf{Vert}(\mathbb{D})(a, d) \end{array}$$

is a (strict) pullback of categories.

**Proposition 4.16.** *Any regular cover  $f: a \twoheadrightarrow b$  in a virtual equipment  $\mathbb{D}$  is a strong cover.*

*Proof.* Suppose we have an inclusion  $g: c \rightarrowtail d$  and a commutative square  $v \circ f = g \circ u$  in  $\mathbf{Vert}(\mathbb{D})$ . We need to show there is a unique arrow  $h: b \rightarrow c$  such that  $g \circ h = v$  and  $h \circ f = u$ . Because  $g$  is an inclusion, there is a unique 2-cell  $\phi$  satisfying

$$\begin{array}{ccc} a & \xrightarrow{\ker(f)} & a \\ f \downarrow & \text{cart} & \downarrow f \\ b & \xrightarrow{b} & b \\ v \downarrow & \Downarrow v & \downarrow v \\ d & \xrightarrow{d} & d \end{array} = \begin{array}{ccc} a & \xrightarrow{\ker(f)} & a \\ u \downarrow & \Downarrow \phi & \downarrow u \\ c & \xrightarrow{c} & c \\ g \downarrow & \Downarrow g & \downarrow g \\ d & \xrightarrow{d} & d \end{array} \quad (16)$$

and it is not hard to check, again using that  $g$  is an inclusion, that  $(u, \phi)$  is an embedding  $\ker(f) \rightarrow c$ . Then, because  $f$  is a regular cover, there is a unique arrow  $h: b \rightarrow c$  satisfying

$$\begin{array}{ccc} a & \xrightarrow{\ker(f)} & a \\ u \downarrow & \Downarrow \phi & \downarrow u \\ c & \xrightarrow{c} & c \end{array} = \begin{array}{ccc} a & \xrightarrow{\ker(f)} & a \\ f \downarrow & \text{cart} & \downarrow f \\ b & \xrightarrow{b} & b \\ h \downarrow & \Downarrow h & \downarrow h \\ c & \xrightarrow{c} & c \end{array} \quad (17)$$

We can read  $h \circ f = u$  directly off (17), while  $g \circ h = v$  follows because it becomes true after precomposition with  $f$ .

If  $h'$  is another arrow such that  $h' \circ f = u$  and  $g \circ h' = v$ , then (17) with  $h'$  in place of  $h$  holds because it becomes true after postcomposition with  $g$ , and therefore  $h' = h$  by the universality of  $f$ .

For the 2-dimensional orthogonality, suppose we have 2-cells  $\alpha: u \Rightarrow u'$  in  $\mathbf{Vert}(\mathbb{D})(a, c)$  and  $\beta: v \Rightarrow v'$  in  $\mathbf{Vert}(\mathbb{D})(b, d)$ , such that  $g \circ \alpha = \beta \circ f$ . Similarly to

(16), there is a unique 2-cell  $\psi$  satisfying

$$\begin{array}{ccc}
 a & \xrightarrow{\ker(f)} & a \\
 f \downarrow & \text{cart} & \downarrow f \\
 b & \xrightarrow{b} & b \\
 v' \downarrow & \Downarrow \beta & \downarrow v \\
 d & \xrightarrow{d} & d
 \end{array}
 =
 \begin{array}{ccc}
 a & \xrightarrow{\ker(f)} & a \\
 u' \downarrow & \Downarrow \psi & \downarrow u \\
 c & \xrightarrow{c} & c \\
 g \downarrow & \Downarrow g & \downarrow g \\
 d & \xrightarrow{d} & d
 \end{array}
 \quad (18)$$

and, using once more that  $g$  is an inclusion, one can verify that  $\psi$  is a bimodule embedding  $u'\psi_u: {}_{\ker(f)}\ker(f)_{\ker(f)} \rightarrow c$ , and also that  $\psi \circ e_{\ker(f)} = \alpha$ , where  $e_{\ker(f)}: a \Rightarrow \ker(f)$  is the unit of the monoid  $\ker(f)$ . Because  $f$  is a regular cover, hence  $\ker(f) \rightarrow b$  is a bimodule collapse, there is a unique  $\gamma$  such that

$$\begin{array}{ccc}
 a & \xrightarrow{\ker(f)} & a \\
 u' \downarrow & \Downarrow \psi & \downarrow u \\
 c & \xrightarrow{c} & c
 \end{array}
 =
 \begin{array}{ccc}
 a & \xrightarrow{\ker(f)} & a \\
 f \downarrow & \text{cart} & \downarrow f \\
 b & \xrightarrow{b} & b \\
 h' \downarrow & \Downarrow \gamma & \downarrow h \\
 c & \xrightarrow{c} & c
 \end{array}
 \quad (19)$$

and this  $\gamma$  is the 2-cell  $h \Rightarrow h'$  in  $\mathbf{Vert}(\mathbb{D})(b, c)$  we wanted. By precomposing (19) with the unit  $e_{\ker(f)}$  of the monoid  $\ker(f)$ , we get  $\gamma \circ f = \alpha$ , and  $g \circ \gamma = \beta$  holds because it becomes true after precomposing with the collapse  $\ker(f) \rightarrow b$ .

Finally, verifying that another  $\gamma'$  satisfying  $\gamma' \circ f = \alpha$  and  $g \circ \gamma' = \beta$  must be equal to  $\gamma$  is analogous to the uniqueness of  $h$  above.  $\square$

**Theorem 4.17.** *Let  $\mathbb{D}$  be a regular virtual equipment. There is an orthogonal 2-factorization system  $(\mathcal{E}, \mathcal{M})$  on the vertical 2-category  $\mathbf{Vert}(\mathbb{D})$ , where  $\mathcal{E}$  is the class of regular covers, and  $\mathcal{M}$  is the class of inclusions.*

*Proof.* The orthogonality of these two classes was proven in Proposition 4.16. The factorization is constructed as in (15). The arrow  $i: c \twoheadrightarrow \langle \ker(f) \rangle$  is clearly regular, and that  $\tilde{f}$  is an inclusion follows directly from Lemma 4.14.  $\square$

**Corollary 4.18.** *In a regular virtual equipment, the classes of strong covers and regular covers coincide.*

*Proof.* By Proposition 4.16 we know that every regular cover is a strong cover.

Given a strong cover  $f: c \twoheadrightarrow d$ , by Theorem 4.17 we can factor  $f = \tilde{f} \circ i$  with  $i$  regular cover and  $\tilde{f}$  an inclusion. Because  $\tilde{f} \circ i$  is a strong cover, it follows that  $i$  is a strong cover as well, hence an isomorphism. Thus  $f$  is a regular cover because  $\tilde{f}$  is.  $\square$

## 5 Exact virtual double categories

Recall that a category  $\mathcal{C}$  with finite limits is called *exact* if every internal equivalence relation in  $\mathcal{C}$  is effective.

**Definition 5.1.** Let  $\mathbb{D}$  be a virtual equipment. Say that  $\mathbb{D}$  is *exact* if  $\mathbb{D}$  is regular, and if every monoid and bimodule in  $\mathbb{D}$  is effective. (See 4.7, 4.4.)

**Proposition 5.2.** *For any virtual equipment  $\mathbb{D}$ , the virtual equipment  $\mathbf{Mod}(\mathbb{D})$  is exact.*

*Proof.* We saw in Proposition 4.9 that  $\mathbf{Mod}(\mathbb{D})$  is regular. Additionally, we saw in Example 3.8 that in fact *all* monoids in  $\mathbf{Mod}(\mathbb{D})$  have a collapse, and it is clear from the construction that the collapse cell is cartesian. Hence every monoid in  $\mathbf{Mod}(\mathbb{D})$  is effective.

From the proof of Proposition 4.9, it is clear that any bimodule in  $\mathbf{Mod}(\mathbb{D})$  has a collapse with the same underlying proarrow in  $\mathbb{D}$ , and that the collapse cell is cartesian. Hence every bimodule in  $\mathbf{Mod}(\mathbb{D})$  is effective.  $\square$

*Remark 5.3.* We might hope to say that for any exact category  $\mathcal{C}$ , the virtual equipment  $\mathbf{Rel}(\mathcal{C})$  is exact, extending Proposition 4.8. However, this is not the case. For  $\mathbf{Rel}(\mathcal{C})$  to be exact would mean that every reflexive and transitive relation (not necessarily symmetric) is the kernel pair of some morphism. This would imply that every reflexive and transitive relation *is* symmetric, and this is clearly not true in general.

It appears that exactness for a virtual equipment is a “directed” generalization of exactness for a category. This directedness is essential to the category-like examples, where the elements of a monoid  $M$  become the morphisms in its collapse  $\langle M \rangle$ . Moreover, it is not even possible to *define* what a symmetric monoid in a virtual equipment is without some extra structure.

**Proposition 5.4.** *A virtual equipment  $\mathbb{D}$  is exact if and only if:*

- *every monoid  $M: c \rightrightarrows c$  has a collapse  $(i, \vec{i}): (c, M) \rightarrow \langle M \rangle$  with  $\vec{i}_M$  cartesian, and*
- *for every pair of monoids  $M, N$ , the restriction functor*

$$i_M \mathbf{Res}_{i_N}: \mathbf{HCor}(\mathbb{D})(\langle M \rangle, \langle N \rangle) \rightarrow {}_M \mathbf{Bimod}_N$$

*is an equivalence of categories.*

*Proof.* To begin, suppose  $\mathbb{D}$  satisfies the conditions of the proposition. Clearly, this implies that every monoid and bimodule in  $\mathbb{D}$  is effective, and that every effective monoid has a collapse. The only thing remaining to check is part 2 of Definition 4.7.

Suppose we have a cartesian cell of the form

$$\begin{array}{ccc} c & \xrightarrow{B(g,f)} & c' \\ f \downarrow & \text{cart} & \downarrow g \\ d & \xrightarrow{B} & d' \end{array}$$

Let  $M = \ker(f)$  and  $N = \ker(g)$ , and without loss of generality let  $d = \langle M \rangle$  and  $f = i_M$ , and similarly for  $d'$  and  $g$ . For any vertical arrows  $h: \langle M \rangle \rightarrow x$  and  $h': \langle N \rangle \rightarrow x'$  and any proarrow  $P: x \rightarrowtail x'$ , we have a string of bijections

$$\begin{aligned} \text{hi}_M \text{Emb}_{h' i_N}(B(i_N, i_M), P) &\cong {}_M \text{Bimod}_N(B(i_N, i_M), P(h' i_N, h i_M)) \\ &\cong \mathbb{D}(B, P(h', h)) \\ &\cong {}_h \mathbb{D}_{h'}(B, P) \end{aligned}$$

where the first is by Lemma 3.3, the second is the second condition of the proposition, and the third is the definition of restriction. This shows that  $B$  is the collapse of  $B(i_N, i_M) = B(g, f)$ .

Conversely, suppose  $\mathbb{D}$  is exact. By assumption, any monoid  $M: c \rightarrowtail c$  is effective, hence  $M$  has a collapse because  $\mathbb{D}$  is regular, and  $\vec{i}_M$  is cartesian by Corollary 4.12.

For the second condition, because  $\mathbb{D}$  is regular we already know from Proposition 4.10 that the restriction functor  $\mathbf{HCor}(\mathbb{D})(\langle M \rangle, \langle N \rangle) \rightarrow {}_M \text{Bimod}_N$  is fully faithful. To see that it is essentially surjective, let  $B \in {}_M \text{Bimod}_N$  be a bimodule. Any  $(M, N)$ -bimodule is effective, so  $B$  has a collapse  $\langle B \rangle: \langle M \rangle \rightarrowtail \langle N \rangle$  by Proposition 4.13, and moreover the embedding  $\vec{i}_B$  is cartesian. Hence  $B \cong {}_{i_M} \text{Res}_{i_N}(\langle B \rangle)$ .  $\square$

In [7, 8], proarrow equipments are introduced as a proposed setting for formal category theory. There the structure of a proarrow equipment was presented in terms of an identity-on-objects pseudo 2-functor  $(-)_*: \mathcal{K} \rightarrow \mathcal{M}$  between bicategories. In [5] it is proven that an equipment (there called a framed bicategory), can be equivalently defined to be a pseudo 2-functor  $(\overline{-}): \mathcal{K} \rightarrow \mathcal{M}$ , where  $\mathcal{K}$  is a strict 2-category and  $\mathcal{M}$  is a bicategory with the same objects,  $(\overline{-})$  is the identity on objects and locally fully-faithful, and such that for every arrow  $f$  in  $\mathcal{K}$ ,  $\bar{f}$  has a right adjoint  $\tilde{f}$  in  $\mathcal{M}$ . This is equivalent to Wood's definition, except that  $\mathcal{K}$  is required to be a strict 2-category.

If  $\mathbb{D}$  is a framed bicategory, then the corresponding proarrow equipment has  $\mathcal{K} = \mathbf{Vert}(\mathbb{D})$  the vertical 2-category and  $\mathcal{M} = \mathbf{HCor}(\mathbb{D})$  the horizontal bicategory of  $\mathbb{D}$ , while for any vertical arrow  $f: c \rightarrow d$ ,  $\bar{f} = d(1, f): \rightarrowtail d$  is the representable proarrow, which has a right adjoint  $\tilde{f} = d(f, 1)$ .

However, in [8] two more axioms are proposed to support the development of formal category theory. The first of these concerns coproducts, which we will not be considering in this paper. The second, there called Axiom 5, concerns Kleisli objects for monads in  $\mathcal{M}$ .

Recall that given a monad  $M: a \rightarrow a$  in a bicategory  $\mathcal{B}$ , a *left  $M$ -module* is an arrow  $X: a \rightarrow b$  together with an action  $X \circ M \Rightarrow X$  satisfying the usual axioms for

monoid action. A homomorphism of left  $M$ -modules from  $X: a \rightarrow b$  to  $X': a \rightarrow b'$  is an arrow  $f: b \rightarrow b'$  and a 2-cell  $f \circ X \Rightarrow X'$  which respects the  $M$  actions in the obvious way. This defines for any monad  $M$  a functor  $\mathbf{LMod}(-, M)$  taking an object  $b$  to the category of left  $M$ -modules  $a \rightarrow b$ . The *Kleisli object* for  $M$  is then defined to be an object  $a_M$  which represents this functor, i.e. equipped with a natural equivalence  $\mathcal{B}(a_M, b) \cong \mathbf{LMod}(b, M)$ . We will refer to the left  $M$ -module  $a \rightarrow a_M$  corresponding to the identity on  $a_M$  as the *universal left  $M$ -module*.

Dually, the *Eilenberg-Moore object*, or EM object,  $a^M$  of  $M$  is a representing object for the functor  $\mathbf{RMod}(-, M)$  sending an object  $b$  to the category of right  $M$ -modules  $b \rightarrow a$ .

If  $(-)_*: \mathcal{K} \rightarrow \mathcal{M}$  is a proarrow equipment, then Wood's Axiom 5 requires that every monad  $M: a \rightarrow a$  in  $\mathcal{M}$  has a representable Kleisli object  $(i_M)_*: a \rightarrow a_M$  such that the adjoint  $(i_M)^*: a_M \rightarrow a$  is an EM object for  $M$ , and such that a composition  $f \circ (i_M)_*$  is representable if and only if  $f$  is.

We will give a slightly strictified version of this axiom, which is appropriate when assuming  $\mathcal{K}$  is a strict 2-category, and then show that this is equivalent to exactness as we defined above. But first we will prepare with a lemma to help translate between the double category formalism and the proarrow equipment formalism

**Lemma 5.5.** *Let  $M: c \rightarrow c$  be a monoid in an equipment  $\mathbb{D}$ . For any 2-cell of the form*

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ f \downarrow & \Downarrow \vec{f} & \downarrow f \\ d & \xrightarrow{d} & d \end{array} \quad (20)$$

*the following are equivalent:*

- $(f, \vec{f})$  is an embedding,
- the corresponding 2-cell  $\vec{f}_*: M \odot d(1, f) \Rightarrow d(1, f)$  in  $\mathbf{HCor}(\mathbb{D})$  is a left  $M$ -action,
- the corresponding 2-cell  $\vec{f}^*: d(f, 1) \odot M \Rightarrow d(f, 1)$  in  $\mathbf{HCor}(\mathbb{D})$  is a right  $M$ -action.

**Definition 5.6.** Let  $\mathbb{D}$  be an equipment. Say that  $\mathbb{D}$  *satisfies Wood's axiom 5* if, for every monoid  $M: c \rightarrow c$ , there is an object  $c_M$ , vertical arrow  $i: c \rightarrow c_M$ , and 2-cell

$$\begin{array}{ccc} c & \xrightarrow{M} & c \\ i \downarrow & \Downarrow \vec{i} & \downarrow i \\ c_M & \xrightarrow{c_M} & c_M \end{array} \quad (21)$$

such that

- the corresponding 2-cell  $\vec{i}_*: M \odot c_M(1, i) \Rightarrow c_M(1, i)$  in  $\mathbf{HCor}(\mathbb{D})$  is a universal left  $M$ -module,
- the corresponding 2-cell  $\vec{i}^*: c_M(i, 1) \odot M \Rightarrow c_M(i, 1)$  in  $\mathbf{HCor}(\mathbb{D})$  is a universal right  $M$ -module, and



- any proarrow  $P: c_M \twoheadrightarrow d$  is representable if (and only if)  $c_M(1, i) \odot P$  is. Moreover, if the latter is represented by  $g: c \rightarrow d$ , then  $P$  is representable by some  $f$  such that  $f \circ i = g$  (an equality, not just an isomorphism).

**Lemma 5.7.** *For any bicategory  $\mathcal{B}$  the following are equivalent:*

1. *Every monad  $M: c \rightarrow c$  in  $\mathcal{B}$  has an object  $c_M$  which is both the Kleisli object and EM object for  $M$ .*
2. *Every monad  $M: c \rightarrow c$  in  $\mathcal{B}$  factors as an adjunction  $i_M \dashv i^M$ ,  $M \cong i^M \circ i_M: c \rightarrow c_M \rightarrow c$ , such that for every pair  $M, N$  of monads the induced functor*

$$\mathcal{B}(c_M, c_N) \xrightarrow{i^N \circ (-) \circ i_M} {}_M\text{Bimod}_N \quad (22)$$

*is an equivalence of categories.*

*Proof.* (1  $\Rightarrow$  2): Let  $M$  be a monad with universal right  $M$ -module  $i^M: c_M \rightarrow c$  and universal left  $M$ -module  $i_M: c \rightarrow c_M$ . It is a standard fact from bicategory theory (see e.g. [6]) that by factoring the unit right  $M$ -module  $M$  through the universal one,  $M \cong i^M \circ \alpha$ , we get an adjunction  $\alpha \dashv i^M$  such that  $M$  is the monad induced by the adjunction. If we similarly factor the unit left  $M$ -module,  $M \cong \beta \circ i_M$ , we get an adjunction  $i_M \dashv \beta$ . It follows that  $\alpha \cong i_M$ ,  $\beta \cong i^M$ , and  $M \cong i^M \circ i_M$ .

To see the equivalence of categories, we only need to note that composition with  $i^N$  induces an equivalence  $\text{LMod}(c_N, M) \cong {}_N\text{Bimod}_M$ , and likewise for  $i_M$ . This is a straightforward check which we leave to the reader. Thus each functor in

$$\mathcal{B}(c_M, c_N) \xrightarrow{(-) \circ i_M} \text{LMod}(c_N, M) \xrightarrow{i^N \circ (-)} {}_M\text{Bimod}_N$$

is an equivalence.

(2  $\Rightarrow$  1): Let  $M: c \rightarrow c$  be a monad. To see that  $i_M$  is a universal left  $M$ -module, simply notice that  $\text{LMod}(b, M) \cong {}_M\text{Bimod}_{1_b}$ . Thus  $i_M$  is a universal left  $M$ -module because

$$\mathcal{B}(c_M, b) \xrightarrow{1_b \circ (-) \circ i_M} {}_M\text{Bimod}_{1_b} \cong \text{LMod}(b, M)$$

is an equivalence of categories. Likewise we can see that  $i^M$  is a universal right  $M$ -module.  $\square$

**Theorem 5.8.** *A framed bicategory  $\mathbb{D}$  is exact if and only if it satisfies Wood's axiom 5.*

*Proof.* **Axiom 5  $\Rightarrow$  exact:** We will use Proposition 5.4 to show that  $\mathbb{D}$  is exact.

Let  $M: c \twoheadrightarrow c$  be a monoid in  $\mathbb{D}$ , and let  $(i, \vec{v}): M \rightarrow c_M$  be the embedding in (21). We wish to show that this is a collapse cell, hence  $c_M \cong \langle M \rangle$ .

Let  $(f, \vec{f}): M \rightarrow x$  be any other embedding. By Lemma 5.5, this makes  $d(1, f)$  a left  $M$ -module, hence there is a unique-up-to-isomorphism proarrow  $X: c_M \twoheadrightarrow d$  such that  $c_M(1, i) \odot X \cong d(1, f)$  and  $\vec{v}_* \odot X = \vec{f}_*$ . By Definition 5.6,  $X$  is representable by

a unique  $\tilde{f}$  such that  $\tilde{f} \circ i = f$ . Finally, under the bijection of Proposition 2.18, the equation  $\vec{i}_* \odot X = \vec{f}_*$  becomes  $\tilde{f} \circ \vec{i} = \vec{f}$ . Thus any embedding  $(f, \vec{f})$  factors uniquely through  $(i, \vec{i})$ , making  $c_M$  the collapse of  $M$ . Moreover, the collapse cell  $\vec{i}$  is cartesian, corresponding to the canonical isomorphism  $M \cong c_M(1, i) \odot c_M(i, 1)$  from Lemma 5.7.

Thus we have shown the first condition of Proposition 5.4, and the second follows directly from Lemma 5.7, hence  $\mathbb{D}$  is exact.

**Exact  $\Rightarrow$  axiom 5:** For every monoid  $M$  we will take the 2-cell (21) to be the collapse cell of  $M$ . That  $\langle M \rangle$  is both the Kleisli and the EM object for  $M$  in  $\mathcal{H}\mathbf{Cor}(\mathbb{D})$  follows from Lemma 5.7 and Proposition 5.4.

For the last bullet of Definition 5.6, let  $X: \langle M \rangle \rightarrowtail d$  be a proarrow, and suppose that  $\langle M \rangle(1, i) \odot X \cong d(1, g)$  for some  $g: c \rightarrow d$ . Then  $d(1, g)$  is a left  $M$ -module, and by Lemma 5.5 this left  $M$  action is equivalent to an embedding  $(g, \vec{g}): M \rightarrow d$ . Factoring this embedding through the collapse  $g = \tilde{g} \circ i_M$  gives an isomorphism  $d(1, g) \cong \langle M \rangle(1, i) \odot d(1, \tilde{g})$  of left  $M$ -modules, and because  $\langle M \rangle(1, i)$  is the universal left  $M$ -module, this implies  $X \cong d(1, \tilde{g})$ .  $\square$

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